



1.6.11 The fundamental theorem of algebra

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Introduction

- In this topic, we will
 - Observe that not all polynomials have real roots
 - Describe the multiplicity of a root of a polynomial
 - Introduce the fundamental theorem of algebra





Review

- The *degree* of a polynomial is the highest power of the indeterminate that has a non-zero coefficient when expanded
 - For example, $\deg(z^7 - 5z + 3) = 7$
 $\deg(z - 5) = 1$
 $\deg(3) = 0$
 $\deg(0) = -\infty$ (special case)
- A polynomial p has a root at a specific point z_0 if $p(z_0) = 0$
 - All non-zero polynomials p have at most $\deg(p)$ roots
 - If we restrict ourselves to real roots, some non-constant polynomials have no roots
 - For example, $z^2 + 1$ and $z^4 - 3z^3 + 4z^2 - z + 2$





Condition for the root of a polynomial

- A polynomial p has a root at a point z_0 if p can be written as

$$p(z) = (z - z_0) q(z)$$

where $q(z)$ is a polynomial of one less degree than p

- For example, $z^3 - 5z^2 + 2z + 8$ has a root at $z = 2$ because

$$z^3 - 5z^2 + 2z + 8 = (z - 2)(z^2 - 3z - 4)$$

- From this point on, roots may be complex:

$$z^4 - z^3 - 3z^2 + 17z - 30 = (z - 1 + 2j)(z^3 - 3j z^2 - (7 - 2j) z + 6 + 12j)$$

- You will not have to find such roots in this course! ☺
- There are software packages that do this





A double root

- A polynomial may have a *double root*
 - For example, z^2 has a double root at $z = 0$
 $z^4 - 3z^3 - 9z^2 + 23z - 12$ has a double root at $z = 1$

- A polynomial p has a double root at z_0 if it can be written

$$p(z) = (z - z_0)^2 q(z)$$

where q is a polynomial of degree two less than the degree of p

- For example, $z^2 = (z - 0)^2 \cdot 1$
 $z^4 - 3z^3 - 9z^2 + 23z - 12 = (z - 1)^2(z^2 - z - 12)$





A root of multiplicity m

- A polynomial p has a root at z_0 of *multiplicity* m if it can be written

$$p(z) = (z - z_0)^m q(z)$$

where q is a polynomial

1. of degree m less than the degree of p
2. where $q(z_0) \neq 0$
 - That is, q does not have a root at z_0





Using polynomial division

- How do we determine the multiplicity of a root z_0 ?
 - Keep dividing the polynomial by $z - z_0$ until the remainder is non-zero
 - Recall that if the remainder after dividing a polynomial p by $z - z_0$ is r , then $p(z_0) = r$
 - Thus, if the remainder is 0, $p(z_0) = 0$





Using polynomial division

- Example:

- Consider the polynomial

$$p(z) = z^4 - 2z^3 - 7z^2 + 20z - 12$$

- We will find the multiplicity of a root (if any) at:
 - $z = 3$
 - $z = 2$
 - $z = 1$





Using polynomial division

- Example:

- Checking at $z = 3$

$$\begin{array}{r} z^3 + z^2 - 4z + 8 \\ z-3 \overline{) z^4 - 2z^3 - 7z^2 + 20z - 12} \\ \underline{z^4 - 3z^3} \\ z^3 - 7z^2 \\ \underline{z^3 - 3z^2} \\ -4z^2 + 20z - 12 \\ \underline{-4z^2 + 12z} \\ 8z - 12 \\ \underline{8z - 24} \\ 12 \end{array}$$

- The polynomial does not have a root at $z = 3$, and at 3, the polynomial evaluates to 12





Using polynomial division

- Example:

- Checking at $z = 2$

$$\begin{array}{r} z^3 \\ z - 2 \overline{) z^4 - 2z^3 - 7z^2 + 20z - 12} \\ \underline{z^4 - 2z^3} \\ - 7z^2 + 20z \\ \underline{- 7z^2 + 14z} \\ 6z - 12 \\ \underline{6z - 12} \\ 0 \end{array}$$

- The polynomial has a root at $z = 2$ of at least multiplicity 1





Using polynomial division

- Example:

- Checking at $z = 2$

$$p(z) = (z - 2)(z^3 - 7z + 6)$$

$$\begin{array}{r} z^2 + 2z - 3 \\ z - 2 \overline{) z^3 } \\ \underline{z^3 - 2z^2} \\ 2z^2 - 7z \\ \underline{2z^2 - 4z} \\ -3z + 6 \\ \underline{-3z + 6} \\ 0 \end{array}$$

- The polynomial has a root at $z = 2$ of at least multiplicity 2





Using polynomial division

- Example:

- Checking at $z = 2$

$$p(z) = (z - 2)^2 (z^2 + 2z - 3)$$

$$\begin{array}{r} z + 4 \\ z - 2 \overline{) z^2 + 2z - 3} \\ \underline{z^2 - 2z} \\ 4z - 3 \\ \underline{4z - 8} \\ 5 \end{array}$$

- The polynomial has a root at $z = 2$ of multiplicity 2
 - The polynomial $z^2 + 2z - 3$ evaluated at $z = 2$ is 5





Using polynomial division

- Example:

- Checking at $z = 1$

$$\begin{array}{r} z^3 - z^2 - 8z + 12 \\ z-1 \overline{) z^4 - 2z^3 - 7z^2 + 20z - 12} \\ \underline{z^4 - z^3} \\ -z^3 - 7z^2 \\ \underline{-z^3 + z^2} \\ -8z^2 + 20z \\ \underline{-8z^2 + 8z} \\ 12z - 12 \\ \underline{12z - 12} \\ 0 \end{array}$$

- The polynomial has a root at $z = 1$
of at least multiplicity 1





Using polynomial division

- Example:

- Checking at $z = 1$

$$p(z) = (z-1)(z^3 - z^2 - 8z + 12)$$

$$\begin{array}{r} z^2 - 8 \\ z-1 \overline{) z^3 - z^2 - 8z + 12} \\ \underline{z^4 - z^3} \\ -8z + 12 \\ \underline{-8z + 8} \\ 4 \end{array}$$

- The polynomial has a root at $z = 1$ of multiplicity 1
 - The polynomial $z^3 - z^2 - 8z + 12$ evaluated at $z = 1$ is 4





Using polynomial division

- Example:

- Consequently, the polynomial

$$p(z) = z^4 - 2z^3 - 7z^2 + 20z - 12$$

has roots at

- $z = 2$ with multiplicity 2
 - $z = 1$ with multiplicity 1
 - The polynomial does not have a root at $z = 3$





Derivative test for multiplicity

- Another test to determine the multiplicity of a root:

- Find

$$p(z_0), \frac{d}{dz} p(z_0), \frac{d^2}{dz^2} p(z_0), \dots$$

until you find the k^{th} derivative that is non-zero at z_0

- In that case, the multiplicity of the root at z_0 is k
 - Remember that $p(z_0)$ can be interpreted as evaluating the zeroth derivative at z_0





Derivative test for multiplicity

- Example:
 - Consider the polynomial

$$p(z) = z^4 - 2z^3 - 7z^2 + 20z - 12$$

$$\frac{d}{dz} p(z) = 4z^3 - 6z^2 - 14z + 20$$

$$\frac{d^2}{dz^2} p(z) = 12z^2 - 12z - 14$$

$$\frac{d^3}{dz^3} p(z) = 24z - 12$$

$$\frac{d^4}{dz^4} p(z) = 24 \longleftarrow \text{The multiplicity cannot exceed 4}$$

$$\underbrace{p(3) = 12}$$

$z = 3$ is not a root

$$\underbrace{p(2) = \frac{d}{dz} p(2) = 0, \quad \frac{d^2}{dz^2} p(2) = 10}$$

$z = 2$ is a root of multiplicity 2

$$\underbrace{p(1) = 0, \quad \frac{d}{dz} p(1) = 4}$$

$z = 1$ is a root of multiplicity 1





Fundamental theorem of algebra

- We have now described multiplicity and described how to find it
 - We now go on to our main result





Fundamental theorem of algebra

The Fundamental Theorem of Algebra:

A polynomial of degree n has exactly n complex roots when counting multiplicity.

Proof (sketch):

The Little Picard Theorem says that every non-constant polynomial must take on every possible complex number for some argument.

A polynomial p of degree $n > 0$ is not constant, and thus, there must be a point z_0 such that $p(z_0) = 0$.

We can use polynomial division to divide out this root so that

$$p(z) = (z - z_0) q(z).$$

As long as $q(z)$ is not constant, we repeat this algorithm on $q(z)$ to find the next root of p .





Examples

- This quintic polynomial

$$p(z) = z^5 - 4z^4 + 3z^3 - z^2 + z - 2$$

has five roots at these points

$$-0.410781594282310 - 0.650948478106310j$$

$$-0.410781594282310 + 0.650948478106310j$$

$$0.844566757558750 - 0.603621101827556j$$

$$0.844566757558750 + 0.603621101827556j$$

$$3.13242967344712$$





Summary

- In this topic, we described the roots of a polynomial
 - A real polynomial of degree n may have fewer than n roots
 - We introduced the concept of multiplicity
 - We gave the fundamental theorem of algebra





References

- [1] https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra
- [2] [https://en.wikipedia.org/wiki/Multiplicity_\(mathematics\)](https://en.wikipedia.org/wiki/Multiplicity_(mathematics))
- [3] https://en.wikipedia.org/wiki/Polynomial_long_division





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Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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for more information.





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